Model of 33-day barotropic Rossby waves in the North Pacific

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Sea-Surface Height (SSH) variability throughout a large part of the North Pacific has been shown to be coherent with Tropical Instability Waves (TIWs). At low latitudes, this variability appears consistent with an interpretation of barotropic Rossby waves radiating poleward from the TIWs, but questions arise regarding the patterns of SSH variability and the ultimate fate of such waves. In this note we analyze the output of a numerical barotropic model of the North Pacific, to address these questions and further test the barotropic Rossby wave interpretation. We find that amplitude patterns in the SSH variability are due partly to topographic refraction, but also to wave interference produced by the broad zonal wavenumber spectrum inherent in the TIW structure. A comparison of output from a model having realistic bathymetry with that of a flat-bottom model suggests that partial trapping associated with topographic refraction increases energy densities and fluxes at low latitudes, while inhibiting the radiation to higher latitudes.
1. Introduction

Farrar (2011) examined Pacific tropical instability waves (TIWs) and associated variability in a gridded satellite altimetry product. He identified coherent sea surface height (SSH) variability extending about 1,000 km northward from the TIWs, and showed that the phase pattern of this variability was consistent with the barotropic Rossby wave dispersion relation on a flat bottom. This was interpreted as evidence that barotropic Rossby waves radiate northward from the instabilities near the equator, a conclusion that is not farfetched given that this phenomenon has been seen in models going back to Cox (1980). It raises the question of what happens to these waves, as they seemed to disappear near 20° N in the observational analysis of Farrar (2011) and the model simulations of Cox (1980), but they seem to go much further in the model simulations of Holmes and Thomas (2016). In a companion paper to this one (Farrar et al., 2020), we re-examined the altimetry record for clearer insights into the fate of these radiating waves. We found that the disappearance of the waves near 20° N is apparently an artifact of how the gridded SSH data product is made, and that the SSH remains coherent with the TIW signal in a region extending several thousand kilometers from the instability. The high coherence, however, occurred only in patches, with regions of high coherence separated by regions of low coherence. Moreover, the amplitude of the SSH variability at TIW frequencies also exhibited marked spatial variability or patchiness, with large variations in amplitude over distances of less than one wavelength.

These new observations raise a number of questions: (1) Can the observed long-range coherence be interpreted as a simple result of barotropic Rossby waves radiating away from the unstable equatorial currents? (2) If so, why would the coherence amplitude drop to insignificant levels and then rise again to high values, and why are there spatial variations in the amplitude of the SSH variability over distances that are small compared to a wavelength? (3) How would realistic
bottom topography and bottom friction modify the flat bottom picture of the wave propagation implicit in the modeling of Cox (1980) and of Holmes and Thomas (2016), and in the analysis of Farrar (2011)?

In this paper, we accept the evidence that the surface intensified TIWs generate or are coupled to barotropic Rossby waves, and we do not address the mechanism responsible for this coupling. We address the above questions concerning the fate of the barotropic waves using a linear barotropic model of the North Pacific, in which Rossby waves are forced at 10° N - the southernmost latitude at which they are clearly identified in the observations. The forcing frequency is central to the frequency band analyzed by Farrar et al. (2020), the phase is consistent with the observed coherence phase near 10° N, and the zonal distribution of the forcing amplitude is consistent with the distribution of the TIW amplitude. One version of the model has realistic bathymetry, and another has a flat bottom.

The TIW amplitude is significantly modulated over a small number of wavelengths in the east-west direction, which implies the existence of a broad spectrum of zonal wavenumbers. The associated wave interference of the outgoing waves produces spatial beats in energy flux and energy density, with parallel ridges of high SSH variability extending into the interior of the flat bottom model. Similar ridges of SSH variability can be seen in the model runs with topography, but there are additional patterns produced by topographic refraction. A similarity between the topographic model’s SSH patterns and those observed by Farrar et al. (2020) suggests that the two mechanisms of wave interference and topographic refraction also contribute to the observed SSH variability. A comparison of the evolution in the topographic and flat-bottom models suggests that partial trapping and topographic refraction enhance the variability at lower latitudes and reduce radiation to higher latitudes.
The model is described in Section 2, and a comparison of the observations and the topographic model output is presented in Section 3. Section 4 presents an analysis of the evolution and the stationary states of both the flat-bottom and topographic models, and in Section 5, we summarize the results and discuss possible reasons for discrepancies between the model output and observations. Throughout the paper, any reference to “observations” without attribution implies the Farrar et al. (2020) analyses.

In the presence of both stratification and bottom topography, the familiar flat-bottom vertical modes can be seriously altered (e.g., Rhines, 1970; Hallberg, 1997; LaCasce, 2017). This raises questions about the utility of a single layer barotropic model for the interpretation of observed SSH. In Appendix A, we use a modified version of Rhines’ perturbation formulation to assess the possible extent to which stratification might alter both the vertical structure of the wave’s perturbation pressure and the dispersion relation. We conclude that the qualitative nature of our model results is likely to be robust.

2. Model configuration

The amplitude of the observed SSH variability is small (a few cm), and we solve the linearized, single layer, shallow-water equations with variable layer depth. The numerical scheme uses an Arakawa C grid with $1^\circ \times 1^\circ$ grid spacing, and the basin depths are a subsample of finer resolution ETOPO2 bathymetry. Given the smoothing inherent in the analysis of Farrar et al. (2020), this model gridding should be able to reproduce any features detected in the observations. The coastline is set at the 200$m$ depth, representing the edge of the continental shelf. Small scale channels with steep bathymetry at high latitudes tended to induce grid-scale instabilities that would radiate into the rest of the basin, so the Alaskan Peninsula and Aleutian Island chain were modeled as a smoothed, continuous coastline. To ensure that the model felt the full impact of the Emperor
Seamount Chain, the ridge line depths from the ETOPO2 one-minute bathymetry were inserted into the ridge-line locations of the one-degree topography. The averaging scheme for the Coriolis terms conserves energy globally (e.g., Sadourny, 1975).

The model bathymetry is shown in Fig. 1a, along with selected contours of the ambient potential vorticity, \( f/h \) (\( f \) is the Coriolis parameter, \( h \) is the water depth). Fig. 1b shows the bottom slope magnitude, calculated from central differences at each grid point. The northwest Pacific basin is bordered by steep bathymetry along the North American coast, the Aleutian Island chain, the Emperor Seamount chain, and the Hawaiian Island chain. Steep bathymetry associated with the east Pacific fracture zones penetrates into the basin from the continent; the most prominent such feature being the Mendocino Fracture Zone near 40° N. Much of the rest of the basin has bottom slopes that are \( O(10^{-3}) \), but there are seamounts scattered throughout the basin with localized steeper bathymetry. In each panel of Fig. 1, the white curve encompasses the region where Farrar et al. (2020) find significant coherence with the TIWs, and hence the region on which we concentrate our modeling efforts.

The vertically averaged momentum and continuity equations are

\[
\frac{\partial \mathbf{u}}{\partial t} + f(\theta) \hat{e}_z \times \mathbf{u} = -g \nabla \eta + h^{-1} \nabla \cdot (h A_h \nabla \mathbf{u}) - R\mathbf{u},
\]

\[
\frac{\partial \eta}{\partial t} + \nabla \cdot (h \mathbf{u}) = 0,
\]

where \( \mathbf{u} \) is the horizontal current (with zonal and meridional components \( u \) and \( v \)), \( \eta \) the SSH displacement, \( h(\phi, \theta) \) the variable water depth, \( f \) the latitudinally dependent Coriolis parameter, and \( g \) the gravitational acceleration. Time is \( t \), and the equations are solved in the spherical-shell coordinates \( (\phi, \theta) \) (longitude, latitude), with the local vertical unit vector being \( \hat{e}_z \). The gradients and divergences are strictly horizontal, and a standard Newtonian friction parameterization of the Reynolds stresses is used. Not all of the terms in the stress parameterization are included (see, for
instance, Müller, 2006, eqs. 13.25–13.26 or Weijer et al., 2007, eqs. 4 and 5). At the length scales of interest, however, the omitted terms are small compared to the divergence of the momentum gradient. Following Warren et al. (2002), we parameterize the bottom drag as friction in an Ekman layer:

\[ R = \sqrt{A_v f / 2h}. \]  

(3)

We varied the vertical viscosity \( A_v \) between 0 and \( 10^{-2} m^2 s^{-1} \) (the high end of the range for deep sea viscosities suggested by Johnson [1998]), and found minimal change in the patterns of SSH variability; including little change in the relative amplitudes of recognizable features. The same was true of varying the horizontal viscosity between \( 1,000 m^2 s^{-1} \) and \( 5,000 m^2 s^{-1} \) (see Section 4c). We primarily display results using \( A_h = 2,500 m^2 s^{-1} \) and \( A_v = 1.6 \times 10^{-3} m^2 s^{-1} \). These values of viscosity attenuate the grid-scale instabilities noted above, and the model reaches an essentially stationary oscillatory state that persists for at least two wave periods before we stop the integration. At the length scale of the incident waves \( |\vec{K}_i|^{-1} \) and over slowly varying topography, the dissipation time scale due to the horizontal viscosity is roughly \( (A_h |\vec{K}_i|^2)^{-1} \approx 180 \) days. This time scale decreases with the wavelength, and this likely contributes to an important qualitative difference between output of the topographic model and that of the companion flat bottom model (addressed in Section 4c).

The model was forced by normal velocity \([V_f(\phi, t)]\) at a southern boundary located at \( 10^\circ N \), where the barotropic Rossby wave first becomes clearly distinguishable in the observations from the TIW signal (Farrar, 2011). The upper panel of Fig. b shows the zonal distribution of the real and imaginary parts of \( V_f \) (normalized by \( \max |V_f| \)). This represents a westward propagating sinusoid with a primary zonal wavelength of \( 16^\circ \), which is consistent with the observed coherence phase. The amplitude modulation of \( V_f \) is a smoothed version of the longitudinal dependence of the observed coherence gain at \( 5^\circ N \), the latitude at which the TIW SSH variance is a maximum.
(see Fig. 3, bottom left panel, and Fig. 3 in Farrar et al., 2020). The forcing velocity represents the meridional velocity of a barotropic Rossby wave that has been recently generated by the TIWs. We do not address the mechanism by which the surface-intensified TIWs generate the barotropic Rossby waves, but we assume that to lowest order, the forcing is proportional to the TIW amplitude. The boundary velocity was oscillated with a period $T = 33.5$ days, and the amplitude was increased gradually from zero over the first period. This minimized the amplitude of higher frequency transients generated by the onset of the forcing.

The amplitude modulation envelope is not much longer than the primary wavelength, so the forcing has a zonal wavenumber spectrum with considerable spread about the primary wavenumber (which also would be true for TIW related motions in the real ocean). The lower panel of Fig. b shows the normalized amplitude of the forcing’s Fourier components vs. the zonal wavenumber (normalized by the primary wavenumber). The width of the distribution at the half-amplitude point is about half the value of the primary wavenumber. Red arrows show the normalized group velocity vectors of selected Fourier components, with the abscissa and ordinate representing eastward and northward axes.

A wave absorber was used on the western boundary to circumvent the necessity of modeling a western boundary, and to prevent recirculation of wave energy via spurious coastal Kelvin waves. The model was run until it reached an essentially stationary state throughout the North Pacific for at least two consecutive periods.

The spectrum of wavenumbers in the forcing produces interference patterns in the SSH variance, which would not be expected from consideration of a single wavenumber. To distinguish between this effect and the effects of topography, we duplicated our numerical experiments in a rectangular, flat-bottom basin with a depth of 5500 m, an eastern boundary at $80^\circ$ W, and a northern boundary at $50^\circ$ N. The southern boundary, the wave absorber on the western boundary, the dissipation, and the
normal velocity forcing at the southern boundary remained identical to those in our topographic model.

3. **Comparison of topographic model output with observations**

The top panels of Fig. 3 show the model’s stationary-state SSH amplitude and phase, regressed onto a 33.5 day cycle over days 101 to 168 after onset of the forcing (periods 4 and 5). The distinguishing features of this stationary state had set up by day 67 after onset (end of period 2), with only minor changes in amplitude occurring during the following period. No noticeable changes in either spatial patterns or amplitudes occurred during the final two periods of integration.

The bottom panels show the observed coherent SSH gain and phase relative to 5°N, 130°W, in the 33-34 day period band (from Farrar et al., 2020). This reference location is the point of maximum observed SSH variance in that period band. In the bottom left panel, two contours of the model amplitude are superimposed upon the observed gain pattern. The main patterns of elevated SSH variability in both the model and the observations fall within a large-scale envelope roughly delineated by the white curve in Fig. 1a. The envelope extends to the northwest from the most energetic region of the TIWs to about 35°N, then gradually curves to a more westward direction and terminates to the east of the Emperor Seamount chain. The shape of the envelope is similar to the shape of an envelope of rays that are started at 10°N with only our forcing’s primary wavenumber and frequency, and traced over smoothed large-scale topography (see Fig. 2 in Farrar et al., 2020).

Both the model output and the observations show three parallel phase lines that are fairly robust between 10°N and about 35°N. These trend SE to NW, indicating southwest phase propagation, with a meridional wavenumber that is roughly equal to the zonal wavenumber and a total wavelength of about 11°. The phase lines are parallel to those which Farrar (2011) showed were
consistent with the barotropic Rossby wave dispersion relation over a flat bottom. Poleward of 35° N, the phase pattern is less well defined. This is where the large-scale amplitude envelope bends toward the west, and where the rays traced by Farrar et al. (2020) refract toward the west and converge at the same time.

In spite of the general robustness of the phase lines up to 35° N, irregularities in the lines can be seen on scales of a few degrees, suggesting small scale perturbations on the larger scale wave field. Both the model SSH amplitude and the observed coherence gain also display a patchiness on similar scales. In Section 4, we show that the model patchiness between 10°N and 40°N is due to two effects: wavenumber dispersion and interference, and topographic refraction and trapping.

The forcing modulation envelope is not much wider than the primary zonal wavelength, yielding a spectrum of wave numbers in the forcing, with slightly different phase orientations and energy propagation directions. This also would be true of the observations (see coherence gain and phase patterns in Fig. 3). The model results are fairly insensitive to period, within roughly 32.5 day to 34.5 day forcing, so the finite bandwidth used in the coherence calculations (Farrar et al., 2020) would not be likely to affect the amplitude patterns. These patterns in the model are sensitive, however, to the wavenumber content of the forcing, so the idealized forcing in the model cannot be expected to precisely reproduce the observed patterns. The qualitative similarity between the observed and modeled patterns between 10°N and 40°N, however, does support the interpretation that within this latitude range the observed coherence can be ascribed to barotropic Rossby waves radiating from the TIWs.

The biggest discrepancy between the observations and the model output involves the elevated coherence gain near 44°N, 180°E. The model does show a local maximum at this location, but the relative amplitude of this latter feature is much smaller than that seen in the observations. We suggest two possible reasons for this. The elevated variability seen in the observations is
collocated with a cul-de-sac in the deep basin bathymetry, bounded to the north, east and south by the Aleutian Islands, the Emperor Seamount chain, and the northwest part of the Hawaiian Ridge (Fig. 1a). There are deep channels at the southern and northern ends of the Emperor Seamounts, so the majority of the topographic relief would likely be porous to Rossby waves. There is a small region of closed $f/h$ contours in the center of the cul-de-sac, however (see Fig. 1a). It is possible that this could provide conditions favorable for a deep basin resonance similar to those reported by Weijer et al. (2007) and Weijer (2008), but which our model is unable to reproduce. Model runs with $1/4$ degree grid spacing (used by Weijer, 2008) also did not produce the level of SSH variability seen in the observations at this location, but we cannot rule out a deficiency in the idealized model as the source of the discrepancy. A second possibility that we feel is unlikely but cannot rule out is that the observed variability is locally forced by winds that are coherent with the TIWs through an atmospheric teleconnection.

4. Model analysis

To understand how the energy flows from the forcing region into the interior, and what determines the SSH patterns, it is useful to examine the energy flux patterns in both flat-bottom and topographic models. The energy equation associated with (1)-(2) is

$$\partial_t E \equiv \partial_t \left[ \frac{1}{2} (h|\vec{u}|^2 + g\eta^2) \right] = -\nabla \cdot \vec{S} - \nabla \cdot (hA_h \nabla |\vec{u}|^2/2) - D,$$  \hspace{1cm} (4)

where $D$ is the dissipation:

$$D = hA_h \left( |\nabla u|^2 + |\nabla v|^2 \right) + \sqrt{A_v f/2} |\vec{u}|^2,$$ \hspace{1cm} (5)

and $\vec{S} = gh\eta \vec{u}$  \hspace{1cm} (6)

is the energy flux due to the pressure work.
Part of the geostrophic velocity contribution to $\vec{S}$ is absolutely non-divergent and cannot contribute to the energy evolution (see Durland and Farrar, 2020). At low frequencies ($\partial_{tt}/f^2 \ll 1$), the potentially divergent part of the energy flux is

$$\vec{S}_d = gL_d^2(\eta^2 \hat{e}_z \times \frac{\vec{\beta}}{2} - \eta \nabla \eta_t), \quad (7)$$

where

$$L_d^2 \equiv \frac{gh}{f^2}, \quad (8)$$

and

$$\vec{\beta} \equiv h \nabla \left(\frac{f}{h}\right) \quad (9)$$

are the squared local deformation radius and the $\beta$ vector ($\nabla f$ on a flat bottom). The first term in the parentheses on the right-hand side (rhs) of (7) represents the contribution of the divergent part of the geostrophic velocity, while the second term represents the contribution of the ageostrophic velocity. Alternately, subtracting the nondivergent part directly from (6) provides the equivalent expression

$$\vec{S}_d = gh\eta \vec{\bar{u}} - g^2 \hat{e}_z \times \nabla \left(\frac{h \eta^2}{f^2}\right), \quad (10)$$

which can be evaluated from a snapshot of the model output. In what follows, “energy flux” refers to $\vec{S}_d$, as evaluated by either (7) or (10). In a narrow-banded wavefield, the period average of $\vec{S}_d$ is proportional to the group velocity, and it follows the paths of refracting rays. In a broad-banded wave field like ours, the concept of a unique group velocity is lost, but $\vec{S}_d$ still represents the flux of energy that can contribute to the evolution of the energy density (Durland and Farrar, 2020).

The depth integrated kinetic energy density is $h|\vec{u}|^2/2$, and the potential energy density is $g\eta^2/2$. The second term on the rhs of (4) is the divergence of the diffusive flux of kinetic energy. With our value of $A_h$, this flux in the flat bottom model is roughly two orders of magnitude smaller than $\vec{S}_d$.

When a statistically steady state is reached (the tendency term on the left-hand side vanishes in the period average), the flux divergences are balanced by the dissipation, $D$, in the period average.
Our primary interest is in the SSH variability, and in what follows we will use the shorthand $\langle PE \rangle$ to designate the model’s period averaged potential energy density ($\langle g\eta^2/2 \rangle$, where $\langle \rangle$ indicates a period average). The period averages are meaningful even while the wave field is evolving, because the evolution is gradual over the time scale of the waves (33.5 days/$2\pi = 5.3$ days). When a day is reported for a particular period average, it refers to the central day of the period.

*a. Consequences of broad wavenumber spectrum*

Most of our theoretical understanding of the refraction of barotropic Rossby waves is based on the assumption that the wave amplitude does not change significantly over a wavelength, implying a narrow wavenumber band for which a single group velocity vector can be associated with the energy flux. Smith (1971), for instance, provided an elegant solution for calculating the refraction of short-Rossby-wave energy flux rays on a sphere with variable bathymetry, with no prior assumption about the relative importance of $\nabla h/h$ vs. $\nabla f/f$. His examples were for highly idealized configurations, but the formulation itself is applicable to arbitrary bathymetry, as long as the wave amplitude, the Coriolis parameter and the water depth are all “slowly varying”: they change only gradually over a wavelength. We find that in some cases our model’s energy flux vectors refract in accordance with the Smith predictions, and in some cases they do not. This is not surprising, since the model configuration violates two of the slowly varying assumptions. First, the wavenumber content of the forcing is not narrow banded (Fig. Figure b). Second, in spite of the fact that our area of interest covers the flattest part of the North Pacific basin, there are important topographic features with scales that are not large relative to the dominant wavelength (compare Figs. 1 and 3). Rather than attempt to interpret the model patterns in light of established theory, then, we will primarily present the numerical solutions. Nevertheless, it is worth considering first how some of
the consequences of a broad wavenumber spectrum relate to our notions of group velocity for a slowly varying packet.

Consider a wave field with a single frequency ($\omega$) but a broad spectrum of discrete wavenumbers ($\vec{K}_n$), which is the situation found in our numerical model once the initial transients have dispersed. The SSH is

$$\eta = \sum_{n=1}^{N} A_n \Re e^{i\phi_n},$$  \hspace{1cm} (11)$$

where $\Re$ denotes the real part of, and

$$\phi_n = \vec{K}_n \cdot \vec{x} - \omega t + \alpha_n.$$ \hspace{1cm} (12)$$

Each $A_n$, $\vec{K}_n$ and $\alpha_n$ is a real valued, “slowly varying” function of the horizontal location $\vec{x}$. If there were only a single wavenumber, the period averaged energy flux [using (7)] would be

$$\langle \vec{S}_{dn} \rangle = \frac{g \omega L_d^2}{2} A_n \left[ -\frac{|\vec{\beta}|}{2\omega} \hat{e}_x - \vec{K}_n \right],$$ \hspace{1cm} (13)$$

which is the product of the period averaged energy density and the group velocity for $\omega$ and $\vec{K}_n$.

The coordinate system is right-handed Cartesian ($\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$), and $\hat{e}_y$ is aligned with $\vec{\beta}$.

For the spectrum of wavenumbers (11), the period averaged energy flux is [again using (7)]

$$\langle \vec{S}_d \rangle = \frac{g \omega L_d^2}{2} \sum_{n=1}^{N} A_n \left( -\frac{|\vec{\beta}|}{2\omega} \hat{e}_x - \vec{K}_n \right) \sum_{m=1}^{N} A_m \cos(\phi_n - \phi_m),$$ \hspace{1cm} (14)$$

or alternately,

$$\langle \vec{S}_d \rangle = \frac{g \omega L_d^2}{2} \sum_{n=1}^{N} \left( -\vec{K}_n - \frac{|\vec{\beta}|}{2\omega} \hat{e}_x \right) A_n^2$$
$$+ \frac{g \omega L_d^2}{2} \sum_{n=1}^{N} \sum_{m \neq n} \left( -\vec{K}_n - \frac{|\vec{\beta}|}{2\omega} \hat{e}_x \right) A_n A_m \cos(\phi_n - \phi_m).$$ \hspace{1cm} (15)$$

The expression on the rhs of the first line of (15) is the vector sum of the energy fluxes calculated for each wavenumber individually [as in (13)]. The expression on the second line represents all the interactions between the velocity associated with one wavenumber and the pressure associated
with a different wavenumber. This part of (15) contains spatial modulation terms that ‘beat’ at the
differences between the wavenumbers:

$$\cos(\phi_n - \phi_m) = \cos \left( (\vec{K}_n - \vec{K}_m) \cdot \vec{x} + (\alpha_n - \alpha_m) \right).$$  \hspace{1cm} (16)

When the wavenumber spectrum is narrow banded, the modulation terms are slowly varying relative to the dominant wavelength. When the width of the wavenumber spectrum is the same order as the dominant wavenumber, as in our situation (see Fig. b), the energy flux can be expected to be modulated on scales comparable to the dominant wavelength. This modulation applies to both the magnitude and the direction of the energy flux, so that the refraction of $\vec{S}_d$ by variations in the ambient potential vorticity field is not simply a weighted sum of the refractive effects we would expect for the individual $\vec{S}_{dn}$ associated with the individual wavenumbers.

The top panel of Fig. 4 shows the zonal distribution of the meridional energy flux near the southern boundary of the model, averaged over the first period after the onset of forcing (centered on day 16.8). The dashed black curve is proportional to the squared amplitude of the forcing, and it demonstrates the flux distribution we might expect if the amplitude modulation were slowly varying over a wavelength. The solid black curve is the meridional energy flux in the flat-bottom model, and the wavenumber beats in the distribution are clear. This period average evolves up to about day 28 (central to the average time span), but the locations of the peaks and valleys remain relatively constant for any period average. The red curve is the meridional energy flux in the topographic model. It has the same wavenumber beat pattern as the flat-bottom model, but it also has additional small-scale features caused by topographic refraction. These features increase in amplitude as the topographic model evolves, and longitudinal variability in $L_d^2$ and $\hat{e}_z \times \vec{B}$ in the topographic model also contributes to differences between the red and black curves. As with the

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1i.e., the meridional component of $\vec{S}_d$ at 10.5° N; 1/2 grid spacing north of the boundary where the forcing is applied.
flat-bottom model, however, the beats in the energy flux distribution produced by the forcing’s broad wavenumber spectrum remain evident in the topographic model.

Fig. 4b shows the flat-bottom model’s $PE$ averaged over the first period, in a plan view extending from the southern boundary to $25^\circ$ N. The vectors are the period averaged values of the energy flux, $\vec{S}_d$. The zonal modulation of the flux entering the basin creates ridges and troughs in the $\langle PE \rangle$ that extend into the basin in the direction of the peak energy fluxes at the boundary. Modulation of the incident energy flux direction can also be seen in the southern-most row of vectors, but this is more subtle than the amplitude modulation. The $\langle PE \rangle$ ridges grow in length and amplitude as the wave field evolves, but their location and orientation remain steady.

Fig. 4c shows the same quantities for the topographic model. The scales are the same as in panel b. The $\langle PE \rangle$ ridges seen in the flat-bottom model are also evident in the topographic model, but additional smaller-scale features are present, created by topographic refraction. The smaller scale peaks in $\langle PE \rangle$ are encompassed by energy flux trajectories with an anticyclonic circulation.

b. Energy flux refraction and topographic trapping

Fig. 5 shows the energy flux vectors, period-averaged on central day 67, when both the flat-bottom and topographic models have reached a nearly stationary state between $10^\circ$ N and $30^\circ$ N. Fig. 5a and Fig. 5b show the topographic model output, and Fig. 5c shows the flat-bottom model output. The zonal vector in the northwest corner of each panel is a scale vector representing $1 m^4 s^{-3}$. The vector scale has been increased in panel c because the maximum energy flux in the topographic model is roughly twice the maximum in the flat-bottom model.

In Fig. 5a, the energy flux vectors are superimposed on contours of the water depth. Although it is not universally true, in some places the energy flux vectors appear to be wholly or partially trapped to the depth contours. They refract to follow the contours with shallow water to the right.
This is evident in the two red boxes. Where the refraction produces a tight anticyclonic circulation, there is elevated SSH variability within that circulation. This can be seen in panel b, where the flux vectors are superimposed on the \( \langle PE \rangle \) field. (The boxes in Fig. 5b are collocated with those in Fig. 5a.) Note that where the flux vectors appear to be topographically trapped, they are intensified relative to the flux vectors in the flat-bottom model. The latter are shown in Fig. 5c, superimposed on the \( \langle PE \rangle \) of the flat-bottom model. The intensified energy flux, tightly trapped to the depth contours, implies greater velocities and greater velocity shears. This has consequences that will be addressed in the next section. Plotting the flux vectors on contours of \( f/h \) is not as illuminating as plotting them on the \( h \) contours. Note that in the flat-bottom model, \( f/h \), its gradient and its divergence are all constant on latitude lines, yet the refraction of the energy flux vectors is not constant along latitude lines (Fig. 5c); evidently another consequence of the wave interference produced by the broad wavenumber band.

Together with Fig. 4 and equation (15), Fig. 5 may help explain the topographic model’s sensitivity to the shape of the forcing’s amplitude envelope. The locations of the \( \langle PE \rangle \) ridges at the southern boundary, and the direction of their extension into the basin depend on the wavenumber spectrum of the forcing and the interference patterns that it generates. Anticyclonic refraction of the energy flux by a particular topographic feature will produce the highest level of \( \langle PE \rangle \) if the topographic feature is under a \( \langle PE \rangle \) ridge, rather than a trough. This is a simplistic picture, because the topographic refraction itself modifies the ridges and troughs. The full picture is more complex, but the locations and orientations of the \( \langle PE \rangle \) ridges and troughs at the boundary where they are generated clearly influence the development of the downstream \( \langle PE \rangle \) patches.

Figs. 6 and 7 show regions of modeled and observed gain (from Fig. 3) near the two red boxes seen in Fig. 5. Figure 6 is for the southernmost box, near the forcing region, and Fig. 7 is for the more northerly box. The top panel (a) of each figure shows the modeled gain, while the bottom (b)
shows the observed gain. The appropriate box from Fig. 5 is shown in each panel. In each figure, the modeled gain has been adjusted by a multiplicative factor for optimal comparison (which does not change the difference in the displayed $\log_{10}(\text{gain})$ between adjacent regions of low and high gain).

In Fig. 6b, we see that the observed gain has a local maximum in the same location as the model (panel a), which is apparently caused by the refraction of the energy flux around the end of the shallower ridge protruding from the east into the deeper basin (Fig. 5, panels a and b). In Fig. 7, we see that a local maximum present in the modeled gain that is not present in the observed gain. We also see, however, that while the trough patterns in the model and observations are similar, they are not identical. The trough in the observed gain is slightly farther to the north, encroaching on the region of topography that is creating the peak in the modeled gain.

Within the black box in Fig. 5b, there is another anticyclonic energy flux circulation surrounding elevated SSH variability in the model. An examination of Fig. 5a, however, shows that this circulation is not obviously associated with topographic effects. We conjecture that this is a consequence of wave interference, with the contributing wave patterns having been modified between the forcing region and the region of the black box. We cannot rule out the possibility, however, that this is another topographic effect that we do not understand.

The evidence suggests that the patchiness seen in the model output (and by inference in the observations) is due to an interplay between wave interference produced by the broad wavenumber spectrum of TIW forcing and topographic features that refract the Rossby wave energy flux on scales of a few degrees. $\langle PE \rangle$ ridges created by the former must be collocated with the latter to produce a local maximum in the $\langle PE \rangle$ field. The refraction in turn modifies the $\langle PE \rangle$ ridges, complicating predictions of local maxima further downstream. Nevertheless, the large-scale structures of the $\langle PE \rangle$ ridges can still be seen in the converged model output and in the observations (Fig. 3).
c. Model evolution, and insensitivity to values of friction parameters

The partial topographic trapping produces local enhancement of both velocities and velocity gradients, which results in greater dissipation in the topographic model, relative to the flat-bottom model. The consequence of this can be seen in Fig. 8, which shows the evolution of the $\langle PE \rangle$ for the flat-bottom model in the left column, and for the topographic model in the right column. In the flat bottom model, the energy propagates to the northern boundary, where it reflects and produces an accumulation of SSH variance near the boundary. At day 101, the flat bottom model is still evolving: the reflection from the northern boundary is propagating back toward the southern boundary with an amplitude still large enough to produce standing wave patterns. In the topographic model, by contrast, most of the SSH variance is confined south of 40° N, and a stationary state has been reached by day 101. In fact, it was virtually stationary by day 67, as very little difference in amplitude can be seen anywhere in the North Pacific between day 67 and day 101. At the plotting scale, no difference can be seen over days 101 to 168 (not shown).

The topographic effects appear to trap the majority of the Rossby wave energy below 40° N, and this is not merely a matter of specifying friction coefficients that are too large. Fig. 9 shows the $\langle PE \rangle$ at day 101 for model runs using four different values of friction parameters; from ($A_v = 0, A_h = 1000m^2s^{-1}$) in the top left panel to ($A_v = 0.01m^2s^{-1}, A_h = 5000m^2s^{-1}$) in the bottom right panel. Although small scale features are enhanced by decreasing the value of $A_h$, the important features of the SSH patterns are essentially the same for all the values of friction parameters used. The ratio between peak amplitudes at higher latitudes and those at lower latitudes also remains fairly constant. Rather than allowing a larger fraction of the incident energy flux to propagate to higher latitudes, reducing the friction parameters just enhances the velocities and their gradients, so
that the latitudinal distribution of the $\langle PE \rangle$ (Fig. 9) and dissipation (not shown) remain effectively unchanged.

5. Summary/Discussion

A barotropic model of the North Pacific was forced by a normal velocity at the southern boundary, with forcing amplitude, frequency and zonal wavenumber that are consistent with observations of Tropical Instability Waves. This was meant to represent the meridional velocity of a Rossby wave field generated by the TIWs. The model SSH variability was compared with patterns of observed SSH coherence gain and phase relative to TIW variability, as analyzed by Farrar et al. (2020). The large-scale spatial envelope of the model’s SSH response matches that of the observations fairly well. Within this envelope, patchiness in SSH amplitude on scales of a few degrees can be seen in both the model output and the observations, with strong qualitatively similarities. Below $40^\circ$ N, the phase patterns in the observations and model output are similar. This suggests that it is reasonable to interpret the observed SSH amplitude patterns as resulting from barotropic Rossby wave radiation away from the Tropical Instability Waves, even where these patterns do not look like what we might expect from a simple radiating wave field.

The SSH patchiness in the model (and by inference, in the observations) is shown to derive from two interacting effects:

1. The TIW amplitude (and hence model forcing) is modulated on a scale not much longer than the zonal wavelength. The Rossby waves radiating away from the TIWs are thus forced by a broad wavenumber spectrum. Interference between components with different wavenumbers produces parallel, stationary ridges in SSH variability ($\langle PE \rangle$) on the scale of the TIW wavelength.
2. Partial topographic trapping enhances velocities and their gradients near steeper topography. Curvature of the topographic contours that produces anticyclonic circulation of the Rossby wave energy flux results in enhanced SSH variability within the curved contours.

Patches of elevated SSH variability in the converged model are seen where \( \langle PE \rangle \) ridges produced by the wave interference cross topographic features capable of producing the anticyclonic refraction. The partial topographic trapping appears to limit the propagation of energy flux to higher latitudes; an effect that is independent of friction parameters.

There are discrepancies between the model output and the observations, and there are several reasons why this might be so. Although there is strong evidence that barotropic Rossby waves are generated at a low latitude by Tropical Instability Waves (Farrar, 2011), we do not address the actual forcing mechanism in this study. Our assumption that the Rossby wave forcing will have a zonal dependence of amplitude and phase similar to that of the TIW is a reasonable lowest order approximation, but may not be precisely correct. We experimented with a range of patterns for the forcing amplitude and phase, and also with a range of forcing latitudes. Adjusting the forcing pattern could make certain hot spots of model variability align better with those in the observations, but at the expense of making others align less well. Although the bathymetry is the same for all experiments, the SSH ridges generated by wave interference shift with the forcing structure, and emphasize different topographic features. We did not find a forcing configuration that produced an exact match between the hot spots produced by the model and those seen in the observations, but the qualitative nature of the SSH patchiness in the model output bore a strong resemblance to that seen in the observations for a wide range of forcing configurations.

Another likely source of model-observations discrepancy is the combined effect of topography and stratification. In Appendix A, we show that within our region of interest, the large-scale bathy-
metric slopes are gradual enough that we might expect perturbations to the large-scale barotropic wave’s SSH signal and dispersion relation, but not $O(1)$ changes. These perturbations could subtly affect both the refraction of the waves and the relative SSH amplitude patterns. Near steeper bathymetry, such as around isolated seamounts, the combined effect of topography and stratification might be more severe, leading to significant discrepancies between the barotropic model’s SSH field and the observations.

The largest discrepancy between the model output and the observations is found near $44^\circ$ N, $180^\circ$ E, where the observations show a relatively high amplitude. The model produces a small local maximum in the region, but it does not have the large relative amplitude seen in the observations. It is possible that a different forcing pattern, with different phase relations between the wavenumber components, could produce more constructive interference at this long distance from the generation region. None of the many different forcing configurations that we experimented with, however, showed a tendency to produce a more elevated response at this location.

If the above discrepancy is due to a model deficiency, a more likely culprit is the model resolution. This particular hot spot is found in a what at first glance appears to be a topographic cul-de-sac south of the Aleutian Islands, east of the Emperor Seamount chain, and north of the Hawaiian Ridge. It is about 15 degrees longitude by 15 degrees latitude in extent (see Fig. 1), and the possibility of a basin resonance seems worth considering. Because the Emperor Seamount Chain is fairly narrow, we extracted the ridge-line depths from the ETOPO2 one-minute bathymetry and inserted them into the ridge-line locations of our one-degree bathymetry, to ensure that the model felt the full impact of the seamount chain. This had little effect on the model response, likely because the seamount chain has deep channels at the south and north ends that are evident even in the 1-minute ETOPO bathymetry. The apparent cul-de-sac is likely to be leaky to Rossby waves and not capable of supporting a resonant response. Within this region, however, there is a smaller
deep area surrounded by a closed $f/h$ contour, and this could conceivably support a deep-basin resonance similar to those seen by Weijer et al. (2007) in the Argentine Basin, and by Warren et al. (2002) and Weijer (2008) in the Mascarene Basin. Reducing our model grid spacing to 1/4 degree did not produce such a resonance, but it is conceivable that resolution greater than we have yet tried could produce such a result. Another intriguing possibility is that wind forcing near 44° N that is coherent with the TIWs could be exciting the SSH response seen at this latitude in the observations. Both this and the deep-basin resonance possibility are under continued investigation.

With the above caveats in mind, we feel that the qualitative match between the model output and the observations supports the conclusion that the high coherence with TIWs of the SSH field throughout much of the North Pacific can be attributed to radiating Rossby waves. The overall envelope of elevated coherence gain can be explained by the refraction of the wave field by the large-scale bathymetry. Below 40° N, the observed SSH patchiness on scales of a few degrees is qualitatively similar to the effects of wave interference and topographic refraction that are seen in the model. Perhaps the most remarkable result from the model and observational analyses is that, in spite of the topographic effects, the large scale wave can be seen extending thousands of kilometers from its source, in approximate agreement with the simplest flat-bottom theory of short barotropic Rossby waves.

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**APPENDIX**

**Influence of stratification on the barotropic mode**
In our barotropic model, the current and pressure are assumed to be depth independent. In reality, the combination of stratification and topography can alter the vertical structure and the dispersion relation of the lowest vertical mode to the extent that it is no longer recognizable as a barotropic mode. In this section, we will refer to the lowest vertical mode as “barotropic,” and use an approach pioneered by Rhines (1970) to assess likely limits on the extent to which actual conditions in the North Pacific might cause this to differ from the classical depth-independent mode (and consequently, the extent to which our model SSH signals might differ from the observations).

Rhines (1970) looked at the combined effect of stratification and topography on low-frequency solutions of the linearized primitive equations in a domain with a uniform bottom slope. When the fractional change in bottom depth ($h$) is small over a characteristic length scale, $L$, the independent variables and the bottom boundary condition can be expanded in a series based on the perturbation parameter

$$
\delta = L |\nabla h|/h_0 \ll 1,
$$

(A1)

where $h_0$ is a characteristic depth, and the gradient operator is strictly horizontal.

Rhines restricted $L$ (and hence $\delta$) by considering a zonal channel of width $L$, with a bottom slope only in the cross-channel, or meridional direction. LeBlond and Mysak (1978) followed Rhines, keeping the strictly meridional bottom slope but removing the channel walls, allowing for a continuous, rather than discrete selection of meridional wavenumbers. Each solution is for a single frequency-wavenumber combination, which brings up the problem that such a solution cannot be valid over the infinite domain on which a single meridional wavenumber exists. LeBlond and Mysak argued for considering a localized response within the open domain, and we follow this conceptual approach. The TIW forcing is significant over only a few wavelengths, and the response is likewise restricted in space, without being constrained by boundaries (see Fig. 3). This implies a spectrum of wavenumbers, and we can conceptually consider a Rhines-type solution for
each wavenumber, where the solution only needs to be valid within the restricted domain where
the individual wavenumber components interfere constructively. We consider a response that is
localized zonally as well as meridionally, and take the formulation of LeBlond and Mysak one
step further by allowing the direction of $\nabla h$ to be arbitrary.

The mathematical formulation of the perturbation problem is presented at the end of this ap-
pendix, but we first summarize the consequences in a less rigorous form. As in flat-bottom
quasigeostrophy, the $O(0)$ currents are geostrophic, and the $O(1)$ dependent variables are elim-
inated from the $O(1)$ equations through substitution of the continuity equation into the curl of
the momentum equations. The resulting equation for the lowest order pressure is separable,
$p^{(0)}(x,y,z) = P(x,y)Z(z)$, with separation constant $\lambda$ (units of velocity$^{-2}$), yielding the familiar
equations for the vertical structure and dispersion relation:

$$\frac{d}{dz}\left(\frac{1}{N^2} \frac{dZ}{dz}\right) = -\lambda Z, \quad (A2)$$

$$\omega = -\frac{\beta k}{(|\vec{K}|^2 + f_0^2\lambda)}. \quad (A3)$$

$N$ is the $z$-dependent buoyancy frequency, the time dependence is $e^{-i\omega t}$, $\vec{K}$ is the vector wavenumber
with zonal component $k$, and $\beta$ is the meridional gradient of the Coriolis parameter. The
bottom boundary condition ($w = -\vec{u} \cdot \nabla h$) for the $O(1)$ equations contains only the interaction of
the geostrophic currents with the bottom slope at $z = -h_0$, and is thus separable:

$$-i\omega \frac{1}{N^2} \frac{\partial p^{(0)}}{\partial z} = \left(\hat{e}_z \times \frac{\nabla p^{(0)}}{f_0}\right) \cdot \nabla h, \quad (A4)$$

or

$$\frac{1}{Z} \frac{dZ}{dz} = -\frac{N^2}{\omega f_0} \hat{e}_z \cdot \vec{K} \times \nabla h \text{ at } z = -h_0, \quad (A5)$$

where $\hat{e}_z$ is the unit vertical vector. For simplicity, we use the rigid-lid boundary condition:

$$\frac{dZ}{dz} = 0 \text{ at } z = 0. \quad (A6)$$
Some of the limiting behaviors (e.g., $\beta \to 0$, $N^2 \to 0$) are not evident from the above equations because the mathematical formulation is for conditions where the influence of $\beta$ is at least comparable to vortex stretching by the bottom slope, and the Burger number, $(Nh_0/f_0L)^2$, is $O(1)$ relative to $\delta$. Under these conditions, though, we can make some general conclusions based on examination of the above equations even for an arbitrary stratification (as long as $N^2 > 0$ everywhere).

With a flat bottom, the eigenvalue for the lowest vertical mode solution is $\lambda = 0$. $dZ/dz = 0$ throughout the water column and (A3) is the short Rossby wave dispersion relation. These statements are also true for a sloping bottom when $\vec{K}$ is parallel (or antiparallel) to $\nabla h$, in which case the geostrophic currents are along isobaths, and are not affected by the bottom slope.

When $\vec{K} \times \nabla h \neq 0$, $dZ/dz$ is non-zero. We are interested in a vertical mode with no zero crossings, however, and (A2) and (A6) imply that under this condition $dZ/dz$ also has no zero crossings. $Z(z)$ is monotonic on $-h_0 < z < 0$: the greatest difference in wave perturbation pressure between any two points in the water column is between the surface and the bottom. In addition, the sign of $\lambda$ is set by the sign of the bottom boundary condition. If we follow the convention that $Z(z) > 0$ on $-h_0 \leq z \leq 0$, then

$$\text{sign}(\lambda) = \text{sign} \left( \frac{dZ}{dz} \big|_{z=-h_0} \right) = \text{sign} \left( \hat{e}_z \cdot \nabla h \times \vec{K} \right). \quad (A7)$$

When $\vec{K}$ is rotated counter-clockwise from $\nabla h$ (through an acute angle), $\lambda > 0$, the vertical structure is surface intensified, and for a given wavenumber vector, the frequency is decreased relative to the flat-bottom solution. When $\vec{K}$ is rotated clockwise from $\nabla h$, $\lambda < 0$, the vertical structure is bottom intensified, and the frequency is increased for a given wavenumber vector. This is consistent with the results of Rhines (1970), in which $\nabla h$ was strictly meridional, and the topographic vortex stretching directly enhanced or detracted from the planetary beta effect. The effects on
the vertical structure and dispersion relation depend only on the relative orientations of $\vec{k}$ and $\nabla h$, however, regardless of whether $\nabla h$ and $\nabla f$ are in alignment.

**a. Implications for model domain**

The perturbation problem is formulated for a uniform bottom slope on a Cartesian $\beta$ plane, and is thus not appropriate for a single solution throughout the North Pacific basin. Our approach is to solve the vertical problem at each grid point individually, as if the stratification and bottom slope at that point were representative of a larger area. We then use the suite of solutions to assess limits on how the barotropic mode might be altered by the combination of stratification and bottom slope. The individual zonal and meridional bottom slopes that contribute to Fig 1b are first smoothed with a Gaussian smoother with a standard deviation of 2.5 degrees (half-amplitude point of $k_{1/2} \approx 0.5 \text{deg}^{-1}$). This does not attenuate all variability shorter than the scale of the large scale wave seen in the phase patterns of Fig. 3 ($11 \text{deg}/2\pi \approx 2\text{deg}$). The bottom slopes thus used should produce conservative estimates of the impacts on the vertical solution for the large-scale wave. The effect of bottom roughness on the wave is left to the numerical solutions. Stratification at each grid point is derived from the World Ocean Atlas-5 data set.

Figure b shows selected inputs and outputs of the lowest-mode vertical solution in the part of the North Pacific where Farrar (2011) first detected evidence of the barotropic waves: 155°W to 120°W and 10°N to 25°N. Panel (b) shows the perturbation parameter, which remains less than about 0.05 throughout the region, confirming the utility of the perturbation approach. Panel (c) shows the meridional wavenumber, which has been non-dimensionalized by the observed zonal wavenumber at 10°N. Farrar (2011) used the flat-bottom dispersion relation to calculate the meridional wavenumber in the displayed region, and found it to be roughly equal to the observed zonal wavenumber, producing phase lines that were oriented southeast to northwest. The results in
panel (c) show that even with the combined effects of stratification and topography, the meridional wavenumber (and hence the dispersion relation) differs little from Farrar’s estimate. Panel (d) shows the ratio of surface pressure to bottom pressure. This ratio differs from unity by at most 10% throughout the region, except in the immediate vicinity of the Hawaiian Islands. Panel (a) shows what LeBlond and Mysak call the “planetary vorticity factor,” the ratio of the planetary vorticity gradient, $\beta$, to the effect of topographic vorticity stretching, $|(f/h)\nabla h|$. This panel suggest that the planetary vorticity gradient remains dominant throughout the region, although we will see that the topography has a significant influence on both energy flux refraction and SSH variance patterns.

Overall, the results in Fig. b give us confidence in the use of the barotropic model to interpret the SSH observations. The neglect of stratification may lead to discrepancies in modeled SSH and energy flux directions, but these are not expected to be significant. The relative effect of stratification depends on the Burger number, $B^2 = (Nh/fL)^2$, where $N$ is the buoyancy frequency, $h$ the water depth, and $f$ the Coriolis parameter. This number decreases rapidly with latitude outside the domain displayed in Fig. b, as stratification weakens and $f$ increases. Away from the steepest bottom slopes, then, errors in the barotropic model at higher latitudes should be even smaller than those suggested by Fig. b.

b. Perturbation solution

The following derivation is cursory, because the only real difference from that of LeBlond and Mysak (1978) is the arbitrary direction of $\nabla h$ in the bottom boundary condition. Using the density equation and hydrostatic balance to eliminate the perturbation density and vertical velocity, the
linearized primitive equations for oscillations of the form \( \exp(-i\omega t) \) are reduced to

\[
-i\omega u - (f_0 + \beta_0 y)v + \partial_x p = 0 \tag{A8}
\]
\[
-i\omega v + (f_0 + \beta_0 y)u + \partial_y p = 0 \tag{A9}
\]
\[
\partial_x u + \partial_y v - \partial_z \left( N^{-2} \partial_z p \right) = 0 \tag{A10}
\]

The horizontal vector velocity, \( \vec{u} \), in a Cartesian coordinate system \((x, y, z)\), has zonal and meridional components \( u \) and \( v \). \( N \) is the buoyancy frequency, and \( p \) is the pressure divided by the Boussinesq reference density. The latitude dependence of the Coriolis parameter has been linearized in a \( \beta \)-plane approximation, and the values of the parameter and its meridional derivative at the reference latitude are \( f_0 \) and \( \beta_0 \). The rigid lid upper boundary condition is

\[
\partial_z p = 0 \text{ at } z = 0, \tag{A11}
\]

and the lower boundary condition for a sloping bottom of depth \( h(x, y) \), is

\[
\frac{i\omega}{N^2} \partial_\bar{z} p = -\vec{u} \cdot \nabla h \text{ at } z = -h. \tag{A12}
\]

Nondimensional variables (designated by a tilde) are

\[
(\tilde{x}, \tilde{y}) = (x, y)/L, \quad (\tilde{u}, \tilde{v}) = (u, v)/U,
\]
\[
\tilde{p} = p/(ULf_0),
\]
\[
\tilde{z} = z/h_0, \quad \tilde{h} = h/h_0,
\]
\[
\tilde{\omega} = \omega/f_0.
\]
In nondimensional form, (A8)-(A12) are

\[-i\tilde{\omega}\tilde{u} - (1 + \delta \hat{\beta} \tilde{y})\tilde{v} + \partial_{\tilde{x}}\tilde{p} = 0 \quad (A13)\]

\[-i\tilde{\omega}\tilde{v} + (1 + \delta \hat{\beta} \tilde{y})\tilde{u} + \partial_{\tilde{y}}\tilde{p} = 0 \quad (A14)\]

\[\partial_{\tilde{x}}\tilde{u} + \partial_{\tilde{y}}\tilde{v} - \partial_{\tilde{z}}(B^{-2}\partial_{\tilde{z}}\tilde{p}) = 0 \quad (A15)\]

\[\partial_{\tilde{z}}\tilde{p} = 0 \text{ at } \tilde{z} = 0, \quad (A16)\]

\[i\tilde{\omega}\partial_{\tilde{z}}\tilde{p} = -\delta B^2 \tilde{u} \cdot \hat{e}_h \text{ at } \tilde{z} = -1) \cdot \hat{e}_h). \quad (A17)\]

The small parameter \(\delta\) is defined in (A1), the Burger number is \(B^2(\tilde{z}) = N^2 h_0 / f_0 L\), and the “beta factor” (as it is called by LeBlond and Mysak) is

\[\hat{\beta} \equiv \frac{\beta_0 h_0}{f_0 |\nabla h|} = \frac{\beta_0 L}{f_0 \delta}. \quad (A18)\]

This term expresses the relative influences of the planetary vorticity gradient and the vortex stretching by the bottom slope, and it is assumed to be \(O(1)\). The unit horizontal vector \(\hat{e}_h\) points in the direction of \(\nabla h\). The variables in (A13)-(A17) are \(O(1)\), with the exception of \(\tilde{\omega}\), which is \(O(\delta)\) (i.e., \(\omega / f_0 \ll 1\)). The dependent variables can thus be expanded in the perturbation series

\[(\tilde{u}, \tilde{v}, \tilde{p}) = \sum_{n=0}^{\infty} \left(u^{(n)}, v^{(n)}, p^{(n)}\right) \delta^n, \quad (A19)\]

\[\tilde{\omega} = \sum_{n=1}^{\infty} \omega^{(n)} \delta^n. \quad (A20)\]

The \(O(0)\) velocities are geostrophic:

\[\tilde{u}^{(0)} = \hat{e}_z \times \nabla p^{(0)}, \quad (A21)\]
and the $O(1)$ equations reduce to a separable equation for $p^{(0)}$:

\[-i\omega^{(1)} \hat{\nabla}^2 p^{(0)} + \hat{\beta} \partial_{\tilde{x}} p^{(0)} = i\omega^{(1)} \partial_{\tilde{z}} \left( B^{-2} \partial_{\tilde{z}} p^{(0)} \right) = -i\omega^{(1)} \tilde{\lambda} P(\tilde{x}, \tilde{y}) Z(\tilde{z}), \]

(A22)

where $p^{(0)}(\tilde{x}, \tilde{y}, \tilde{z}) = P(\tilde{x}, \tilde{y}) Z(\tilde{z})$, and $\tilde{\lambda}$ is the non-dimensional separation constant.

The horizontal equation yields a wave solution of the form $\exp(\vec{\tilde{k}} \cdot \vec{\tilde{x}})$, with dispersion relation

\[\omega^{(1)} = -\hat{\beta} \tilde{k}/ \left( |\vec{\tilde{k}}|^2 + \tilde{\lambda} \right), \]

(A23)

where $\tilde{k}$ is the zonal component of the non-dimensional vector wavenumber $\vec{\tilde{k}}$. The vertical equation and its boundary conditions are

\[\frac{d}{d\tilde{z}} \left( B^{-2} \frac{dZ}{d\tilde{z}} \right) = -\tilde{\lambda} Z, \]

(A24)

\[\frac{dZ}{d\tilde{z}} = 0 \text{ at } \tilde{z} = 0 \]

(A25)

\[\frac{dZ}{d\tilde{z}} = -\frac{B^2}{\omega^{(1)}} Z \hat{\varepsilon}_{\tilde{z}} \cdot \hat{\varepsilon}_k \times \hat{\varepsilon}_h \text{ at } \tilde{z} = -1. \]

(A26)

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Fig. 1. Panel a: Model bathymetry, $h$, in meters (color) and selected contours of ambient potential vorticity ($Q = f/h$). Panel b: Bottom slope, $|\nabla h|$, based on central differences. In each panel, the white curve encompasses the region where Farrar et al. (2020) find significance of coherence with the TIWs.

Fig. 2. Top panel: Zonal distribution of the real (black) and imaginary (red) parts of meridional velocity imposed on the southern boundary of the barotropic model ($V_f$). The dashed curve is the amplitude modulation envelope, which is a smoothed version of the longitudinal dependence at $5^\circ$ N of the observed coherence gain seen in Fig. 3. Bottom panel: Normalized amplitudes of $V_f$’s Fourier coefficients (circles) vs. zonal wavenumber. The red arrows are normalized group velocity vectors of selected Fourier components, with the abscissa and ordinate representing eastward and northward axes.

Fig. 3. Comparison of model output with observations (a reproduction of Figs. 7 and 8 in Farrar et al. [2020]). Top left: $\log_{10}$ of model SSH amplitude. Bottom left: $\log_{10}$ of observed gain (amplitude of coherent variability) relative to $5^\circ$ N, $130^\circ$ W (black circle), with contours of model SSH amplitude overlaid (black contour = $10^{-1.5}$ and pink contour = $10^{-1.7}$). Top right: Phase of modeled 33-day SSH signal. Bottom right: Observed phase relative to the 33-day SSH signal at $5^\circ$N, $130^\circ$W.

Fig. 4. Panel a: zonal distribution of meridional energy flux at $10.5^\circ$ N, averaged over first period after onset of forcing. Black curve: flat bottom model; red curve: topographic model; black dashed curve: proportional to the square of the forcing amplitude modulation. Panel b: $\log_{10}$ of flat-bottom model’s $PE$ ($m^3 s^{-2}$), averaged over first period after onset of forcing. Panel c: $\log_{10}$ of topographic model’s $PE$, averaged over first period after onset of forcing. Vectors in panels b and c show the energy flux ($\vec{S_d}$), averaged over first period after onset of forcing. The magnitude of the long zonal vector in the northwest corner of each panel is $1 m^4 s^{-3}$.

Fig. 5. Period averaged energy flux vectors and $\langle PE \rangle$ at day 67. Top panel: Topographic model’s energy flux vectors ($\vec{S_d}$) superimposed on contours of $h$ (m). Middle panel: Topographic model’s energy flux vectors superimposed on contours of $\langle PE \rangle$ ($m^3 s^{-2}$). Bottom panel: Flat bottom model’s energy flux vectors superimposed on contours of $\langle PE \rangle$ ($m^3 s^{-2}$). The zonal arrow in the upper left corner of each panel shows the scale for $1 m^4 s^{-3}$. The vector scale is increased in the bottom panel because the maximum energy flux in the flat bottom model is about half the maximum flux in the topographic model, because of intensification associated with the topographic trapping.

Fig. 6. $\log_{10}$ of modeled (panel a) and observed (panel b) gain, extracted from Fig. 3, and centered around the southermost red box seen in Fig. 5, panel a or b. The red box in each panel is collocated with the red box from Fig. 5. The modeled gain has been adjusted by a multiplicative factor for optimal comparison (which does not change the difference in the displayed $\log_{10}$(gain) between adjacent regions of low and high gain).

Fig. 7. Same as Fig. 6, except for the more northerly red box seen in Fig. 5, panel a or b.

Fig. 8. Evolution of flat-bottom (left column) and topographic (right column) models. $\log_{10}$ of the period averaged $\langle PE \rangle$ ($m^3 s^{-2}$) is displayed, with each average centered on the integration day displayed to the left of the columns. The flat-bottom model’s domain is rectangular, and the North American coastline is merely superimposed for ease of comparison with the topographic model output. The color scale for Day 17 is different than for the other days, because the forcing amplitude is being gradually increased during the first period.
**Fig. 9.** Insensitivity of topographic model’s $\langle PE \rangle$ patterns to a range of vertical ($A_v$) and horizontal ($A_h$) friction coefficients. Color contours are $\log_{10}$ of the topographic model’s $\langle PE \rangle$ ($m^3s^{-2}$), with the period average centered on day 101.5 (3 wave periods) after onset of the forcing. The $\langle PE \rangle$ patterns essentially have reached a stationary state by this time for all of the friction values displayed. The magnitude of the $\langle PE \rangle$ decreases with increasing friction coefficients. The color scales have been adjusted to emphasize the similarity of the patterns, and of the ratio of high latitude-to-low latitude $\langle PE \rangle$.

**Fig. A1.** Panel a: $\log_{10}$ of the planetary vorticity factor, $\hat{\beta} = \beta / \left[ (f/h) |\nabla h| \right]$. Panel b: the perturbation parameter, $\delta = \max \left[ \langle L/h \rangle \hat{\nabla} h \cdot \hat{e}_\phi, \langle L/h \rangle \hat{\nabla} h \cdot \hat{e}_\theta \right]$. Panel c: nondimensional meridional wavenumber. Panel d: ratio of surface pressure to bottom pressure.
FIG. 1. Panel a: Model bathymetry, \( h \), in meters (color) and selected contours of ambient potential vorticity \( Q = f/h \). Panel b: Bottom slope, \(|\nabla h|\), based on central differences. In each panel, the white curve encompasses the region where Farrar et al. (2020) find significance coherence with the TIWs.
FIG. 2. Top panel: Zonal distribution of the real (black) and imaginary (red) parts of meridional velocity imposed on the southern boundary of the barotropic model \( V_f \). The dashed curve is the amplitude modulation envelope, which is a smoothed version of the longitudinal dependence at 5° N of the observed coherence gain seen in Fig. 3. Bottom panel: Normalized amplitudes of \( V_f \)’s Fourier coefficients (circles) vs. zonal wavenumber. The red arrows are normalized group velocity vectors of selected Fourier components, with the abscissa and ordinate representing eastward and northward axes.
Fig. 3. Comparison of model output with observations (a reproduction of Figs. 7 and 8 in Farrar et al. [2020]).

Top left: Log_{10} of model SSH amplitude. Bottom left: Log_{10} of observed gain (amplitude of coherent variability) relative to 5°N, 130°W (black circle), with contours of model SSH amplitude overlaid (black contour = 10^{-1.5} and pink contour = 10^{-1.7}). Top right: Phase of modeled 33-day SSH signal. Bottom right: Observed phase relative to the 33-day SSH signal at 5°N, 130°W.
FIG. 4. Panel a: zonal distribution of meridional energy flux at 10.5° N, averaged over first period after onset of forcing. Black curve: flat bottom model; red curve: topographic model; black dashed curve: proportional to the square of the forcing amplitude modulation. Panel b: \( \log_{10} \) of flat-bottom model’s \( PE \) (\( m^3 s^{-2} \)), averaged over first period after onset of forcing. Panel c: \( \log_{10} \) of topographic model’s \( PE \), averaged over first period after onset of forcing. Vectors in panels b and c show the energy flux (\( \vec{S}_d \)), averaged over first period after onset of forcing. The magnitude of the long zonal vector in the northwest corner of each panel is \( 1 m^4 s^{-3} \).
FIG. 5. Period averaged energy flux vectors and $\langle PE \rangle$ at day 67. Top panel: Topographic model’s energy flux vectors ($\vec{S}_d$) superimposed on contours of $h$ (m). Middle panel: Topographic model’s energy flux vectors superimposed on contours of $\langle PE \rangle$ ($m^3 s^{-2}$). Bottom panel: Flat bottom model’s energy flux vectors superimposed on contours of $\langle PE \rangle$ ($m^3 s^{-2}$). The zonal arrow in the upper left corner of each panel shows the scale for $1 m^4 s^{-3}$. The vector scale is increased in the bottom panel because the maximum energy flux in the flat bottom model is about half the maximum flux in the topographic model, because of intensification associated with the topographic trapping.
FIG. 6. $\log_{10}$ of modeled (panel a) and observed (panel b) gain, extracted from Fig. 3, and centered around the southernmost red box seen in Fig. 5, panel a or b. The red box in each panel is collocated with the red box from Fig. 5. The modeled gain has been adjusted by a multiplicative factor for optimal comparison (which does not change the difference in the displayed $\log_{10}(\text{gain})$ between adjacent regions of low and high gain).
Fig. 7. Same as Fig. 6, except for the more northerly red box seen in Fig. 5, panel a or b.
FIG. 8. Evolution of flat-bottom (left column) and topographic (right column) models. $\log_{10}$ of the period averaged $\langle PE \rangle$ ($m^3 s^{-2}$) is displayed, with each average centered on the integration day displayed to the left of the columns. The flat-bottom model’s domain is rectangular, and the North American coastline is merely superimposed for ease of comparison with the topographic model output. The color scale for Day 17 is different than for the other days, because the forcing amplitude is being gradually increased during the first period.
Fig. 9. Insensitivity of topographic model’s ⟨PE⟩ patterns to a range of vertical (Av) and horizontal (Ah) friction coefficients. Color contours are log10 of the topographic model’s ⟨PE⟩ (m³s⁻²), with the period average centered on day 101.5 (3 wave periods) after onset of the forcing. The ⟨PE⟩ patterns essentially have reached a stationary state by this time for all of the friction values displayed. The magnitude of the ⟨PE⟩ decreases with increasing friction coefficients. The color scales have been adjusted to emphasize the similarity of the patterns, and of the ratio of high latitude-to-low latitude ⟨PE⟩.
Fig. A1. Panel a: log$_{10}$ of the planetary vorticity factor, $\hat{\beta} = \beta / \left[ (f/h) |\tilde{\nabla} h| \right]$. Panel b: the perturbation parameter, $\delta = \max \left[ (L/h) \tilde{\nabla} h \cdot \hat{e}_\theta, (L/h) \tilde{\nabla} h \cdot \hat{e}_\phi \right]$. Panel c: nondimensional meridional wavenumber. Panel d: ratio of surface pressure to bottom pressure.